Unitarity of Weyl-Invariant New Massive Gravity and Generation of Graviton Mass via Symmetry Breaking

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We give a detailed analysis of the particle spectrum and the perturbative unitarity of the recently introduced Weyl-invariant version of the new massive gravity in 2+1 dimensions. By computing the action up to second order in the fluctuations of the metric, the gauge and the scalar fields around the anti-de Sitter (AdS) and flat vacua, we find that the theory describes unitary (tachyon and ghost-free) massive spin-2, massive (or massless) spin-1 and massless spin-0 excitations for certain ranges of the dimensionless parameters. The theory is not unitary in de Sitter space. Scale invariance is either broken spontaneously (in AdS background) or radiatively (in flat background) and hence the masses of the particles are generated either spontaneously or at the second loop order.

I. INTRODUCTION

Einstein's general relativity (GR) is expected to be modified at both large (astrophysical) (IR) and small (UV) regions. There are ample theoretical (in the case of UV) and experimental (in the case of IR) reasons to conclude that GR can only be an effective theory that works perfectly in the intermediate regions, such as the solar system and *etc.* Apriori, the nature of UV and IR modifications is quite different. For UV modifications, experience from quantum field theory dictates that if one is to define a perturbatively well-behaved (that is renormalizable and unitary) gravity theory, then one must introduce higher powers of curvature that modify both the tree-level propagator structure and the interactions. Unfortunately, it is well-known that such a theory simply does not exist in four dimensions [1]. On the other extreme, IR modifications consist of introducing a cosmological constant and/or mass to the graviton. Even though, theoretically, cosmological Einstein theory is the easiest extension of GR, the problems with the cosmological constant are well-known (such as the difficulty of keeping it small in the quantum theory). Graviton mass on the other hand is a very subtle issue. Given a massless free spin-2 field about a maximally symmetric background, one can introduce the Fierz-Pauli term respecting the background symmetries to get a massive spin-2 field. But such a theory does not seem to arise from a diffeomorphism invariant interacting gravity theory save the unique case of the 2+1 dimensions.

For $D = 2 + 1$ dimensions, new massive gravity (NMG) introduced in [2] provides a non-linear extension of the Fierz-Pauli massive spin-2 theory. For the mostly plus signature the action reads[1]

$$I_{NMG} = \frac{1}{\kappa^2} \int d^3x \sqrt{-g}\Big[\sigma R - 2\lambda m^2 + \frac{1}{m^2}\Big(R_{\mu\nu}^2 - \frac{3}{8}R^2\Big)\Big],$$
(1)

¹ To have a maximally symmetric vacuum one must have $\lambda > $ -1 and one can normalize $\sigma^2 = 1$ in NMG. On the other hand, $\lambda = 0$ and σ should be free in the Weyl-invariant version, since the numerical values of various couplings play a key role in the unitarity analysis.

which attracted a lot of attention: Detailed works on it appeared in [2–9] regarding its unitarity, solutions and *etc.* Remarkably, higher curvature terms in this theory provide in some sense both the viable UV and IR modifications that one is interested in. Unfortunately, this state of happy affairs do not extend to four dimensions. But in any case, 2+1 dimensional gravity is a valuable theoretical lab for ideas in quantum gravity. [In fact, according to the proposal of Horava for which spacetime's spectral dimension reduces at high energies, 3D gravity becomes much more relevant and NMG appears as part of the non-covariant 3+1 dimensional action [10].]

Having understood that NMG describes a consistent parity-invariant[2] massive spin-2 theory in 2+1 dimensions, the next natural question is to ask if graviton mass can be generated from breaking a symmetry (not diffeomorphism invariance) in this theory in analogy with the Higgs mechanism in the Standard Model. This question was answered in the affirmative recently in [12] by finding a local scale-invariant (Weyl-invariant) version of NMG and showing that in the case of (A)dS space, the vacuum breaks the conformal symmetry spontaneously and for flat space conformal symmetry is broken at the two loop level [13] via the Coleman-Weinberg mechanism [14]. Referring to [12] for the details of how the Weyl-invariant extension of NMG (and other higher curvature models, such as the Born-Infeld NMG [15]) was introduced and how symmetry gets broken, here we just quote the final expression

$$
\begin{aligned}
S_{WNMG} = \int d^3x \sqrt{-g} \Big\{ & \sigma \Phi^2 \big(R - 4\nabla \cdot A - 2A^2 \big) \\
& + \Phi^{-2} \Big[R_{\mu\nu}^2 - \frac{3}{8} R^2 - 2R^{\mu\nu}\nabla_\mu A_\nu + 2R^{\mu\nu} A_\mu A_\nu \\
& + R\,\nabla \cdot A - \frac{1}{2} R A^2 + 2F_{\mu\nu}^2 + (\nabla_\mu A_\nu)^2 \\
& - 2A_\mu A_\nu \nabla^\mu A^\nu - (\nabla \cdot A)^2 + \frac{1}{2} A^4 \Big] \Big\} + S_\Phi + S_{A_\mu},
\end{aligned}
\tag{2}
$$

where S_Φ and S_{A_μ} are the Weyl-invariant scalar and gauge field actions, which are given by

$$
\begin{aligned}
S_\Phi &= -\frac{1}{2} \int d^3x \sqrt{-g} \Big\{ \Big(\partial_\mu \Phi - \frac{1}{2} A_\mu \Phi \Big)^2 + \nu \Phi^6 \Big\}, \\
S_{A_\mu} &= \beta \int d^3x \sqrt{-g}\, \Phi^{-2} F_{\mu\nu}^2.
\end{aligned}
\tag{3}
$$

The action (2) is invariant under the simultaneous transformation of the metric and the fields as

$$
g_{\mu\nu} \to g'_{\mu\nu} = e^{2\zeta(x)} g_{\mu\nu}, \qquad \Phi \to \Phi' = e^{-\frac{(n-2)}{2}\zeta(x)}\Phi, \qquad A_\mu \to A'_\mu = A_\mu - \partial_\mu \zeta(x). \tag{4}
$$

It is important to note that there are no dimensionful parameters in the theory, on the other hand, local scale invariance does not fix the relative numerical coefficients of various parts which are independently scale-invariant. Generically, up to a numerical scaling of the total action, there are 4 dimensionless parameters that one can introduce. By scaling the total action, we set the coefficient of the kinetic part of the scalar action to its canonical non-ghost form and to keep contact with the NMG, we take the numerical coefficient of the quadratic part of the action to be 1 (this choice can easily be relaxed). Therefore, we have 3 dimensionless parameters σ, ν, β.

The Weyl-invariant theory (2) obviously is much larger than NMG (1) in the sense that when one sets $\Phi = \sqrt{m}$, $\nu = 2\lambda$ and $A_\mu = 0$, at the level of the action, one recovers NMG (1), with a

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[2] Parity non-invariant massive spin-2 theory with a single helicity degree of freedom, that is the Topologically Massive Gravity, was found in 1982 [11].

fixed gravitational coupling $\kappa = m^{-1/2}$ and a fixed cosmological. [In fact all the dimensionful scales are determined by the symmetry breaking order parameter $< \Phi >= \Phi = \sqrt{m}$]. Let us consider the infinite dimensional field space $\mathcal{M} = [g_{\mu\nu}, A_\mu, \Phi]$ to be the space of all fields satisfying the field equations derived from the action (2). It was shown in [12] that "NMG-point", that is $[g_{\mu\nu}, 0, \sqrt{m}]$, is in $\mathcal{M}$. Moreover, if one freezes the scalar and the gauge fields to these NMG-point values and consider fluctuations just in the metric directions, one exactly gets the same spectrum as NMG around its AdS or flat vacua. This was shown in [12] but what was left out in that work and which will be remedied here, is a complete study of the second order fluctuations of all the fields around the NMG point (or the vacuum of the theory). Namely, apriori, the stability and unitarity of (2) is not clear for all allowed fluctuations in the metric, gauge and scalar field directions on $\mathcal{M}$. The main task of this paper is to show that NMG-point is stable by proving that there are no ghosts and tachyons in the particle spectrum of the Weyl-invariant action (2). Therefore, mass of the graviton and the mass of the gauge field is consistently generated by the symmetry breaking mechanism of the conformal symmetry.

The layout of the paper is as follows: In section II, we first find the expansion of the action up to second order in the fields around the (A)dS or flat vacua. This section also discusses issues about Jordan versus Einstein frame and the Weyl-invariant gauge-fixing in the gauge sector. In section III, we decouple the fields and identify the masses and also the unitarity regions of the dimensionless parameters. We collect some useful computations in the appendices.

II. QUADRATIC FLUCTUATIONS ABOUT THE VACUUM

In [12], the field equations coming from the action (2) and its vacuum solution were given. Here, we do not depict the field equations, since they are rather lengthy, instead we note that the vacuum solution (let us first take a dS or AdS vacuum, as the flat vacuum will follow these) is given as

$$\Phi_{vac} = \sqrt{m}, \qquad A_{vac}^\mu = 0, \qquad g_{\mu\nu} = \bar{g}_{\mu\nu}, \tag{5}$$

here $\bar{R}_{\mu\nu} = 2\Lambda \bar{g}_{\mu\nu}$. And the cosmological constant satisfies[3]

$$\Lambda^2 + 4\sigma m^2 - \nu m^4 = 0. \tag{6}$$

Given m^2, generically, there are two vacua

$$\Lambda_\pm = m^2 \Big[-2\sigma \pm \sqrt{4\sigma^2 + \nu} \Big]. \tag{7}$$

Our task now is to study the stability of these vacua and also study the particle spectrum of the model. This can be achieved by considering the second order fluctuations about the vacuum following from

$$\Phi = \sqrt{m} + \tau \Phi_L, \qquad A_\mu = \tau A_\mu^L, \qquad g_{\mu\nu} = \bar{g}_{\mu\nu} + \tau h_{\mu\nu}, \tag{8}$$

where we have introduced τ, a small dimensionless parameter to keep the track of the expansion orders. In what follows, we will use the conventions given in [16]. The expansions of various curvature terms are needed in the computations, so we collect them in Appendix A.

Since the action (2) is highly complicated with fields coupled to each other, it is a non-trivial task to find the basic oscillators (free particles) of the theory. There are a couple of paths one can take.

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[3] Here we take the point of view that Φ_{vac} is given and Λ is determined, one can also take a different point of view that Λ is given and Φ_{vac} is determined. See [12] for a discussion on this.

For example, one can linearize field equations and try to decouple the fields. Or, one can transform the action to the Einstein frame and then find the field equations and do the linearization. These two paths do not give an efficient way for the study of the spectrum. [See Appendix B for the Einstein frame version of the Weyl-invariant quadratic theory.] As a third way, one can directly compute the action up to quadratic order in the fluctuations about its vacua, which we shall adopt here. This will lead to coupled fields at the quadratic level. Then we will find a way to decouple the basic free fields in the theory. This procedure is quite lengthy but there seems to be no way of avoiding it and it is still easier than the above mentioned procedures. The action (2), after making use of the field fluctuations (8) and the relevant formulas in the Appendix A, can be written as

$$S_{WNMG} = \bar{S}_{WNMG} + \tau S^{(1)}_{WNMG} + \tau^2 S^{(2)}_{WNMG} + \mathcal{O}(\tau^3), \tag{9}$$

where $\bar{S}_{WNMG}$ is the value of action evaluated in the background which is irrelevant for our purposes. On the other hand $S^{(1)}_{WNMG}$ vanishes in the vacuum, which also gives us the vacuum equations without going into the details of finding the full field equations [12]. Finally the quadratic part $S^{(2)}_{WNMG}$, after making use of the vacuum equations and dropping the boundary terms, reads as

$$
\begin{aligned}
S^{(2)}_{WNMG} = \int d^3x \sqrt{-g} \Big\{ & -\frac{1}{2}(\partial_\mu \Phi^L)^2 + \Big(6\sigma\Lambda - \frac{9\Lambda^2}{2m^2} - \frac{15\nu m^2}{2}\Big)\Phi_L^2 \\
& + \frac{2\beta+5}{2m}(F^L_{\mu\nu})^2 - \Big(2\sigma m + \frac{\Lambda}{m} + \frac{m}{8}\Big)A_L^2 - \frac{1}{m}(\bar{\nabla}\cdot A^L)^2 \\
& + \frac{1}{m}(\mathcal{G}^L_{\mu\nu})^2 - \Big(\frac{\sigma m}{2} - \frac{\Lambda}{4m}\Big)h^{\mu\nu}\mathcal{G}^L_{\mu\nu} - \frac{1}{8m}R_L^2 \\
& + \Big(2\sigma\sqrt{m} + \frac{\Lambda}{m\sqrt{m}}\Big)\Phi^L R^L - \Big(8\sigma\sqrt{m} + \frac{4\Lambda}{m\sqrt{m}} + \frac{\sqrt{m}}{2}\Big)\Phi^L \bar{\nabla}\cdot A^L \Big\}.
\end{aligned}
\tag{10}
$$

In deriving the above expansion, in addition to the formulas in the Appendix, the following relations have also been used

$$\Phi^2 = m\Big(1 + 2\tau\frac{\Phi_L}{\sqrt{m}} + \tau^2\frac{\Phi_L^2}{m} + \mathcal{O}(\tau^3)\Big), \qquad \Big(\nabla_\mu A_\nu\Big) = \tau\bar{\nabla}_\mu A^L_\nu - \tau^2\Big(\Gamma^\gamma_{\mu\nu}\Big)_L A^L_\gamma + \mathcal{O}(\tau^3). \tag{11}$$

The first thing to observe is that to have a non-ghost and canonically normalized (that is $-\frac{1}{4}$) kinetic term for the Maxwell field, we should set $\beta = -\frac{11}{4}$, which we do from now on. As it stands, the fields are still coupled and one should find a way to decouple them. Such a coupling between the scalar field and the curvature is expected, since we are dealing with a non-minimally (in fact conformally) coupled scalar field to gravity. The scalar field also couples to the gauge field as demanded by conformal invariance. To understand how one could decouple these fields in (10), let us study a simpler model (scalar-tensor theory) first, and then come back to our problem.

A. Quadratic fluctuations and the spectrum of the conformally coupled scalar-tensor theory

We choose the 2+1 dimensional conformally coupled scalar-tensor action which is given by

$$S_{S-T} = \int d^3x \sqrt{-g}\Big(\Phi^2 R + 8\partial_\mu\Phi\partial^\mu\Phi - \frac{\nu}{2}\Phi^6\Big), \tag{12}$$

and ask what the particle spectrum is around its (A)dS vacuum. By just inspecting the action, one mistakingly namely think that the scalar field is a ghost since it comes with a negative kinetic energy part. But this is actually a red-herring, since the action is in the Jordan frame one cannot

draw such a conclusion from the full non-linear theory. One must either go to the Einstein frame where the fundamental degrees of freedom are more transparent or, in the Jordan frame, study the quadratic fluctuations of the fields around the vacuum. We will do both below.

Under the conformal rescaling $g_{\mu\nu}(x) = \Omega^{-2}(x)g_{\mu\nu}^E(x)$, with $\Omega \equiv (\frac{\Phi}{\Phi_0})^2$, the action (12) transforms into the Einstein frame, as

$$^E S_{S-T} = \int d^3x \sqrt{-g^E} \Phi_0^2 \Big(R^E - \frac{\nu}{2}\Phi_0^4 \Big), \tag{13}$$

in which Φ_0 is a constant and introduced in order to keep Ω dimensionless. Therefore, the conformally-coupled scalar field simply disappears and one is left with pure cosmological Einstein theory which has a massless spin-2 particle in its spectrum. How does one see this result in the Jordan frame (which we need for our main problem). We take (12) and expand up to quadratic order in the scalar and tensor fields about the (A)dS vacuum to get

$$
S_{S-T} = \int d^3x \sqrt{-\bar g} \Big\{ 6m\Lambda - \frac{\nu}{2}m^3 + \tau \Big[(3m\Lambda - \frac{\nu}{4})h + (12\sqrt{m} - 3\nu m^{5/2})\Phi^L + m\,R^L \Big]
$$
$$
+ \tau^2 \Big[(-\frac{1}{2}m\Lambda + \frac{\nu}{8}m^3)h_{\mu\nu}^2 - \frac{1}{2}mh^{\mu\nu}\mathcal{G}_{\mu\nu}^L + (\frac{1}{4}m\Lambda - \frac{\nu}{16}m^3)h^2 \tag{14}
$$
$$
+ 2\sqrt{m}R^L\,\Phi^L + (6\Lambda - \frac{15}{2}\nu m^2)\Phi_L^2 + 8(\partial_\mu \Phi_L)(\partial^\mu \Phi_L) \Big] \Big\}.
$$

Again $\mathcal{O}(\tau^0)$ part is not relevant. $\mathcal{O}(\tau^1)$ part gives the vacuum of the theory, inserting the value of R_L (Appendix) in the linear part and dropping the boundary terms, one obtains

$$\Lambda = \frac{\nu m^2}{4}. \tag{15}$$

Using this value in the quadratic part results in

$$S_{S-T}^{(2)} = \int d^3x \sqrt{-\bar g} \Big\{ -\frac{1}{2}mh^{\mu\nu}\mathcal{G}_{\mu\nu}^L + 2\sqrt{m}R_L\Phi_L - 24\Lambda\Phi_L^2 + 8(\partial_\mu\Phi_L)^2 \Big\}. \tag{16}$$

By the redefinition of tensor field as follows

$$h_{\mu\nu} \equiv \widetilde h_{\mu\nu} - \frac{4}{\sqrt{m}}\bar g_{\mu\nu}\Phi_L, \tag{17}$$

(16) reduces to the linearized version of the cosmological Einstein theory

$$S_{S-T}^{(2)} = -\frac{1}{2}m\int d^3x \sqrt{-\bar g}\, \widetilde h^{\mu\nu}\widetilde{\mathcal{G}}_{\mu\nu}^L. \tag{18}$$

As it is clear, just like in the Einstein frame, here at the quadratic level of the Jordan frame, the conformally-coupled scalar field with the wrong-sign kinetic energy disappears in (18). We will use a similar field redefinition in (10).

B. Weyl-invariant gauge-fixing condition

Before we can identify the fundamental degrees of freedom, there is one more issue that we must discuss: The gauge field in its locally Lorentz invariant form, has spurious (non-propagating) degrees of freedom, which we must eliminate. This can be done with a Weyl-invariant gauge-fixing.

Such a gauge condition can be found as follows: Let the gauge-covariant derivative act on the gauge field as [12]

$$\mathcal{D}_\mu A_\nu \equiv \nabla_\mu A_\nu + A_\mu A_\nu. \tag{19}$$

Under the transformations (4), in n dimensions, it is easy to show that the divergence transforms as

$$(\mathcal{D}_\mu A^\mu)' = e^{-2\zeta}\Big(\mathcal{D}_\mu A^\mu - \mathcal{D}_\mu \partial^\mu \zeta + (n-3)(A^\alpha \partial_\alpha \zeta - \partial_\alpha \zeta \partial^\alpha \zeta)\Big), \tag{20}$$

so in 3 dimensions by setting $\mathcal{D}_\mu \partial^\mu \zeta = 0$, we have

$$(\mathcal{D}_\mu A^\mu)' = e^{-2\zeta}(\mathcal{D}_\mu A^\mu). \tag{21}$$

Therefore, we can choose a Lorenz-like condition

$$\mathcal{D}_\mu A^\mu = \nabla \cdot A + A^2 = 0, \tag{22}$$

as a Weyl-invariant gauge-fixing condition. It is important to note that $\mathcal{D}_\mu \partial^\mu \zeta = 0$ is also Weyl-invariant. [This is a Weyl-invariant generalization of the leftover gauge-invariance, $\partial^2 \zeta = 0$, after the usual Lorenz gauge $\partial_\mu A^\mu = 0$ is chosen.] At the linear level, (22) reduces to the background covariant Lorenz condition: $\bar{\nabla} \cdot A_L = 0$.

These tools are sufficient to decouple the fundamental degrees of freedom in (10) which we do in the next section.

III. PARTICLE SPECTRUM AND THEIR MASSES

The Weyl-invariant gauge-fixing term (22) at the linear level eliminates the cross term between the gauge field and scalar field in (10). On the other hand, redefinition (17) works well in decoupling scalar and tensor fields, so at the end (10) becomes

$$\widetilde{S}_{WNMG} = \int d^3 x \sqrt{-\bar{g}} \Big\{ -\frac{1}{2}\Big(16\sigma + \frac{8\Lambda}{m^2} + 1\Big)(\partial_\mu \Phi^L)^2$$
$$- \frac{1}{4m}(F_{\mu\nu}^L)^2 - \Big(2\sigma m + \frac{\Lambda}{m} + \frac{m}{8}\Big)(A_\mu^L)^2 \tag{23}$$
$$- \Big(\frac{\sigma m}{2} - \frac{\Lambda}{4m}\Big)\tilde{h}^{\mu\nu}\widetilde{\mathcal{G}}_{\mu\nu}^L + \frac{1}{m}(\widetilde{\mathcal{G}}_{\mu\nu}^L)^2 - \frac{1}{8m}\widetilde{R}_L^2 \Big\},$$

where we have used the following relations that arise after the field redefinition of (17)

$$(R_{\mu\nu})_L = (\widetilde{R}_{\mu\nu})_L + \frac{2}{\sqrt{m}}(\bar{\nabla}_\mu \partial_\nu \Phi_L + \bar{g}_{\mu\nu}\Box \Phi_L), \quad R_L = \widetilde{R}_L + \frac{8}{\sqrt{m}}(\Box \Phi_L + 3\Lambda \Phi_L),$$

$$\mathcal{G}_{\mu\nu}^L = \widetilde{\mathcal{G}}_{\mu\nu}^L + \frac{2}{\sqrt{m}}\Big(\bar{\nabla}_\mu \partial_\nu \Phi_L - \bar{g}_{\mu\nu}\Box \Phi_L - 2\Lambda \bar{g}_{\mu\nu}\Phi_L\Big),$$

$$h^{\mu\nu}\mathcal{G}_{\mu\nu}^L = \tilde{h}^{\mu\nu}\widetilde{\mathcal{G}}_{\mu\nu}^L + \frac{4}{\sqrt{m}}\widetilde{R}_L \Phi_L + \frac{16}{m}\Phi_L \Box \Phi_L + \frac{48}{m}\Lambda \Phi_L^2, \tag{24}$$

$$(\mathcal{G}_{\mu\nu}^L)^2 = (\widetilde{\mathcal{G}}_{\mu\nu}^L)^2 + \frac{8}{m}(\Box \Phi_L)^2 + \frac{40}{m}\Lambda \Phi_L \Box \Phi_L + \frac{48}{m}\Lambda^2 \Phi_L^2 + \frac{2}{\sqrt{m}}\widetilde{R}_L \Box \Phi_L + \frac{4}{\sqrt{m}}\Lambda \widetilde{R}_L \Phi_L.$$

The expression (23) is what we were looking for to identify the fundamental excitations and their masses. The first line shows that we have a unitary massless scalar field as long as we have a non-ghost kinetic term which is guaranteed by

$$16\sigma + \frac{8\Lambda}{m^2} + 1 \geq 0. \tag{25}$$

In fact, when the bound is saturated the scalar field ceases to be dynamical. The second line is the action for a massive spin-1 field (a Proca field) which propagates 2 unitary degrees of freedom in 2+1 dimensions with mass-square

$$M_A^2 = (4\sigma + \frac{1}{4})m^2 + 2\Lambda \geq 0, \tag{26}$$

which is exactly equal to the constraint on the kinetic energy of the scalar field (25). The third line needs a little more explanation, since the fundamental degrees of freedom are not transparent by a cursory look. But, the action is exactly what one gets from the linearization of the NMG (1) around its (A)dS vacuum [albeit with fixed ratios of dimensionful parameters]. There are two ways to find that it describes a massive spin-2 field. The first way is to show with the help of an auxiliary field that the action reduces to the massive Fierz-Pauli spin-2 theory [2]. The second way is to explicitly decompose the $h_{\mu\nu}$ into its irreducible components and at the end express all the fundamental degrees of freedom as scalar fields [7]. These two pictures yield obviously the same result showing that the third line of (23) describes a massive spin-2 field with mass-square

$$M_g^2 = -\sigma m^2 + \frac{\Lambda}{2}. \tag{27}$$

Unitarity of the massive spin-2 theory depends whether one is dealing with an AdS ($\Lambda < 0$) or a dS ($\Lambda > 0$) background. For the AdS background, Breitenlohner-Freedman (BF) bound [17, 18] $M_g^2 \geq \Lambda$ must be satisfied, on the other hand for the dS background, Higuchi [19] bound $M_g^2 \geq \Lambda > 0$ must be satisfied. What we have not yet shown is that all the unitarity conditions on the spin-0, spin-1 and the spin-2 fields are compatible with each other and with the condition that the theory has a maximally symmetric vacuum (7). First of all one should notice that Λ_+ corresponds to the dS and Λ_- corresponds to the AdS spaces. It is easy to see that (27) is not compatible with the dS branch, therefore the theory is not unitary in dS. On the other hand, considering all the conditions together, one finds that the theory is unitary in AdS with a massless spin-0, a massive spin-1 and a massive spin-2 field as long as[4]

$$-\frac{1}{16} < \sigma \leq 0, \qquad 0 < \nu \leq \frac{1}{64}(1 - 256\sigma^2),$$
$$0 < \sigma \leq \frac{1}{16}, \qquad 0 \leq \nu \leq \frac{1}{64}(1 - 256\sigma^2). \tag{28}$$

On the other hand, for AdS the theory has a massless spin-1 field, a massive spin-2 field (with $M_g^2 = -\frac{3m^2}{16}$) (no scalar field) for

$$\sigma = \frac{1}{16}, \qquad \nu = 0, \qquad \Lambda_- = -\frac{m^2}{4}. \tag{29}$$

For flat vacuum the theory becomes unitary when

$$-\frac{1}{16} \leq \sigma \leq 0, \qquad \nu = 0, \tag{30}$$

for which generically the theory has a massless spin-0, massive spin-1 and massive spin-2 fields. There are two special points: for $\sigma = -\frac{1}{16}$, there is no scalar field, there is a massless gauge field and a massive spin-2 field with mass $M_g = \frac{m}{4}$. For $\sigma = 0$, there is a massless spin-0, a massless spin-2 and a massive spin-1 field with $M_A = \frac{m}{2}$.

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[4] Note that negative ν is not allowed for the scalar field to have a viable potential with a lower bound. Moreover, for $\nu = 0$, the conformal symmetry cannot be broken since Coleman-Weinberg potential at any loop would vanish in the case of flat space, but in what follows below, we include this point to explore the full parameter range.

Conclusions

By computing the action up to second order in all directions in the space of gauge and scalar fields and the metric, we have shown that the Weyl-invariant extension of NMG is a unitary theory generically describing a massive spin-2, a massive (or massless) spin-1 and a massless spin-0 field around its AdS and flat vacua. The mere existence of an AdS background spontaneously breaks the conformal symmetry and provides mass to the spin-1 and spin-2 fields in analogy with the Higgs-mechanism. Breaking of the conformal symmetry also fixes all the relevant couplings between the fields. In flat space, dimensionful parameter (that is the expectation value of the scalar field) comes from dimensional transmutation in the quantum theory and the conformal symmetry is broken at the two loop level [13] via the Coleman-Weinberg mechanism. Weyl-invariant version of NMG seems to be the only known toy model where a graviton mass is generated by the breaking of a symmetry in such a way that the resultant mass has a non-linear, fully covariant, local extension in terms of quadratic curvature terms. In a separate work [20], we study the particle spectrum of similar Weyl-invariant quadratic theories in generic n-dimensions whose actions were introduced in [12]. It would also be interesting to add conformally coupled spin fields to these models.

IV. ACKNOWLEDGMENTS

Appendix A: Quadratic expressions of the curvature terms

This part compiles all the relevant tensors expanded up to second order around a background $(\bar{g}_{\mu\nu})$. We take the expressions directly from [16] for generic n-dimensions. The metric perturbation $h_{\mu\nu}$ is defined as

$$g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \tau h_{\mu\nu}, \qquad g^{\mu\nu} = \bar{g}^{\mu\nu} - \tau h^{\mu\nu} + \tau^2 h^{\mu\rho} h_\rho^\nu + \mathcal{O}(\tau^3). \tag{31}$$

The Christoffel connection can be expanded as

$$\Gamma^\rho_{\mu\nu} = \bar{\Gamma}^\rho_{\mu\nu} + \tau \left(\Gamma^\rho_{\mu\nu}\right)_L - \tau^2 h^\rho_\beta \left(\Gamma^\beta_{\mu\nu}\right)_L + \mathcal{O}(\tau^3), \tag{32}$$

from which follows the expansions of the Riemann and all the related tensors. We just need the following expressions:

$$\left(\Gamma^\rho_{\mu\nu}\right)_L = \frac{1}{2} \bar{g}^{\rho\lambda} \left(\bar{\nabla}_\mu h_{\nu\lambda} + \bar{\nabla}_\nu h_{\mu\lambda} - \bar{\nabla}_\lambda h_{\mu\nu}\right), \tag{33}$$

$$\sqrt{-g} = \sqrt{-\bar{g}} \left[1 + \frac{\tau}{2} h + \frac{\tau^2}{8} \left(h^2 - 2h_{\mu\nu}^2\right) + \mathcal{O}(\tau^3)\right], \tag{34}$$

where $h = \bar{g}^{\mu\nu} h_{\mu\nu}$.

$$R^\mu{}_{\nu\rho\sigma} = \bar{R}^\mu{}_{\nu\rho\sigma} + \tau \left(R^\mu{}_{\nu\rho\sigma}\right)_L - \tau^2 h^\mu_\beta \left(R^\beta{}_{\nu\rho\sigma}\right)_L$$
$$- \tau^2 \bar{g}^{\mu\alpha} \bar{g}_{\beta\gamma} \left[\left(\Gamma^\gamma_{\rho\alpha}\right)_L \left(\Gamma^\beta_{\sigma\nu}\right)_L - \left(\Gamma^\gamma_{\sigma\alpha}\right)_L \left(\Gamma^\beta_{\rho\nu}\right)_L\right] + \mathcal{O}(\tau^3), \tag{35}$$

where the linearized Riemann tensor is defined as follows

$$\left(R^{\mu}{}_{\nu\rho\sigma}\right)_L = \frac{1}{2}\left(\bar{\nabla}_\rho\bar{\nabla}_\sigma h^{\mu}_\nu + \bar{\nabla}_\rho\bar{\nabla}_\nu h^{\mu}_\sigma - \bar{\nabla}_\rho\bar{\nabla}^\mu h_{\sigma\nu} - \bar{\nabla}_\sigma\bar{\nabla}_\rho h^{\mu}_\nu - \bar{\nabla}_\sigma\bar{\nabla}_\nu h^{\mu}_\rho + \bar{\nabla}_\sigma\bar{\nabla}^\mu h_{\rho\nu}\right). \tag{36}$$

The quadratic expansion of the Ricci tensor follows as

$$\begin{aligned} R_{\nu\sigma} = &\bar{R}_{\nu\sigma} + \tau\left(R_{\nu\sigma}\right)_L - \tau^2 h^{\mu}_\beta\left(R^{\beta}{}_{\nu\mu\sigma}\right)_L \\ &- \tau^2 \bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}\left[\left(\Gamma^{\gamma}_{\mu\alpha}\right)_L\left(\Gamma^{\beta}_{\sigma\nu}\right)_L - \left(\Gamma^{\gamma}_{\sigma\alpha}\right)_L\left(\Gamma^{\beta}_{\mu\nu}\right)_L\right] + \mathcal{O}(\tau^3), \end{aligned} \tag{37}$$

where the linearized Ricci tensor is

$$R^L_{\nu\sigma} = \frac{1}{2}\left(\bar{\nabla}_\mu\bar{\nabla}_\sigma h^{\mu}_\nu + \bar{\nabla}_\mu\bar{\nabla}_\nu h^{\mu}_\sigma - \Box h_{\sigma\nu} - \bar{\nabla}_\sigma\bar{\nabla}_\nu h\right). \tag{38}$$

The quadratic expansion of the curvature scalar is

$$\begin{aligned} R = \bar{R} + \tau R_L + \tau^2\Big\{&\bar{R}^{\rho\lambda}h_{\alpha\rho}h^{\alpha}_\lambda - h^{\nu\sigma}\left(R_{\nu\sigma}\right)_L - \bar{g}^{\nu\sigma}h^{\mu}_\beta\left(R^{\beta}{}_{\nu\mu\sigma}\right)_L \\ &- \bar{g}^{\nu\sigma}\bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}\left[\left(\Gamma^{\gamma}_{\mu\alpha}\right)_L\left(\Gamma^{\beta}_{\sigma\nu}\right)_L - \left(\Gamma^{\gamma}_{\sigma\alpha}\right)_L\left(\Gamma^{\beta}_{\mu\nu}\right)_L\right]\Big\} + \mathcal{O}(\tau^3), \end{aligned} \tag{39}$$

where

$$R_L = \bar{g}^{\alpha\beta}R^L_{\alpha\beta} - \bar{R}^{\alpha\beta}h_{\alpha\beta}. \tag{40}$$

The linear form of the Einstein tensor that we frequently used in the text is

$$\mathcal{G}^L_{\mu\nu} = (R_{\mu\nu})^L - \frac{1}{2}\bar{g}_{\mu\nu}R^L - \frac{2\Lambda}{n-2}h_{\mu\nu}. \tag{41}$$

Appendix B: Weyl-invariant action in the Einstein frame

Here we will transform Weyl-invariant new massive gravity (2), which is necessarily in the Jordan frame to the Einstein frame. In what follows we will keep some of the computations in n-dimensions for the sake of generality and set $n = 3$ later. $g_{\mu\nu}(x)$ denotes the Jordan frame metric and $g^E_{\mu\nu}(x)$ denotes the Einstein frame metric which are related as

$$g_{\mu\nu}(x) = \Omega^{-2}(x)g^E_{\mu\nu}(x), \qquad \sqrt{-g} = \Omega^{-n}\sqrt{-g^E}. \tag{42}$$

The Riemann and the Ricci tensors and the curvature scalars in the two frames are related to each other, respectively as follows

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma}[g] = &(R^{\mu}{}_{\nu\rho\sigma})^E - 2\delta^{\mu}_{[\sigma}\nabla_{\rho]}\partial_\nu\ln\Omega - 2g_{\nu[\rho}\nabla_{\sigma]}\partial^\mu\ln\Omega \\ &- 2\partial_{[\sigma}\ln\Omega\,\delta^{\mu}_{\rho]}\partial_\nu\ln\Omega + 2g_{\nu[\sigma}\partial_{\rho]}\ln\Omega\,\partial^\mu\ln\Omega + 2g_{\nu[\rho}\delta^{\mu}_{\sigma]}(\partial_\lambda\ln\Omega)^2. \end{aligned} \tag{43}$$

$$\begin{aligned} R_{\nu\sigma}[g] = &(R_{\nu\sigma})^E + (n-2)\left[\nabla_\sigma\partial_\nu\ln\Omega + \partial_\nu\ln\Omega\,\partial_\sigma\ln\Omega - g^E_{\nu\sigma}(\partial_\lambda\ln\Omega)^2\right] \\ &+ g^E_{\nu\sigma}\Box\ln\Omega. \end{aligned} \tag{44}$$

$$R[g] = \Omega^2\left(R^E + 2(n-1)\Box\ln\Omega - (n-1)(n-2)(\partial_\lambda\ln\Omega)^2\right). \tag{45}$$

From these we can get the invariant relevant for quadratic gravity as :

$$(R_{\mu\nu\rho\sigma})^2[g] = \Omega^4\Big\{(R^E_{\mu\nu\rho\sigma})^2 + 8(R^E_{\mu\nu})\nabla^\mu\partial^\nu\ln\Omega + 8(R^E_{\mu\nu})\partial^\mu\ln\Omega\ \partial^\nu\ln\Omega - 4R^E(\partial_\mu\ln\Omega)^2$$
$$+ 4(n-2)(\nabla_\mu\partial_\nu\ln\Omega)^2 + 4(\Box\ln\Omega)^2 - 8(n-2)\Box\ln\Omega(\partial_\mu\ln\Omega)^2 \tag{46}$$
$$+ 8(n-2)(\partial_\mu\ln\Omega)(\partial_\nu\ln\Omega)\nabla^\mu\partial^\nu\ln\Omega + 2(n-1)(n-2)(\partial_\mu\ln\Omega)^4\Big\},$$

$$(R_{\mu\nu})^2[g] =$$
$$\Omega^4\Big\{(R^E_{\mu\nu})^2 + 2(n-2)(R^E_{\mu\nu})\nabla^\nu\partial^\mu\ln\Omega + 2R^E\Box\ln\Omega + 2(n-2)R^E_{\mu\nu}\partial^\mu\ln\Omega\ \partial^\nu\ln\Omega$$
$$- 2(n-2)R^E(\partial_\mu\ln\Omega)^2 + (n-2)^2(\nabla_\nu\partial_\mu\ln\Omega)^2 + (3n-4)(\Box\ln\Omega)^2 \tag{47}$$
$$+ 2(n-2)^2(\partial_\mu\ln\Omega)(\partial_\nu\ln\Omega)\nabla^\mu\partial^\nu\ln\Omega - (4n-6)(n-2)\Box\ln\Omega(\partial_\mu\ln\Omega)^2$$
$$+ (n-2)^2(n-1)(\partial_\mu\ln\Omega)^4\Big\},$$

$$R^2[g] =$$
$$\Omega^4\Big\{(R^E)^2 + 4(n-1)R^E\Box\ln\Omega - 2(n-1)(n-2)R^E(\partial_\mu\ln\Omega)^2 + 4(n-1)^2(\Box\ln\Omega)^2$$
$$- 4(n-1)^2(n-2)\Box\ln\Omega(\partial_\mu\ln\Omega)^2 + (n-1)^2(n-2)^2(\partial_\mu\ln\Omega)^4\Big\}. \tag{48}$$

As noted in the text above equation (13) one should choose $\Omega \equiv (\frac{\Phi}{\Phi_0})^2$, where Φ_0 is a dimensionful constant which keeps the Einstein-frame metric dimensionless. Note that the mere requirement that there is a transformation between the Jordan and the Einstein frame introduces a dimensionful constant and breaks the scaling symmetry. This symmetry breaking is not the spontaneous symmetry breaking in the vacuum that we discussed in the bulk of the paper. With the help of the above formulas, we can now write the Einstein frame version of the Weyl-invariant quadratic theory (2) (not to clutter the notation, below we will drop the superscript E)

$$\tilde{S}_{NMG} = \int d^3x\sqrt{-g}\Big\{\sigma\Phi_0^2\Big[R - 8(\partial_\mu\ln\Phi)^2 + 8A^\mu\partial_\mu\ln\Phi - 2A^2\Big]$$
$$+ \Phi_0^{-2}\Big[R_{\mu\nu}^2 - \frac{3}{8}R^2 + 4R_{\mu\nu}\nabla^\mu\partial^\nu\ln\Phi + 8R_{\mu\nu}\partial^\mu\ln\Phi\partial^\nu\ln\Phi - 2R\Box\ln\Phi$$
$$- 2R(\nabla_\alpha\ln\Phi)^2 + 4(\nabla_\mu\partial_\nu\ln\Phi)^2 + 8(\nabla_\alpha\ln\Phi)^4 - 4(\Box\ln\Phi)^2$$
$$+ 16(\nabla_\mu\partial_\nu\ln\Phi)\partial^\mu\ln\Phi\partial^\nu\ln\Phi - 8R_{\mu\nu}A^\mu\partial^\nu\ln\Phi$$
$$+ 2RA^\mu\partial_\mu\ln\Phi - \frac{1}{2}RA^2 + 2R_{\mu\nu}A^\mu A^\nu + 4\Box\ln\Phi\nabla\cdot A$$
$$- 8\partial_\mu\ln\Phi\partial_\nu\ln\Phi\nabla^\mu A^\nu - 16A^\mu\partial_\mu\ln\Phi(\partial_\nu\ln\Phi)^2$$
$$- 4\nabla_\mu\partial_\nu\ln\Phi\Big(\nabla^\mu A^\nu + 4A^\mu\partial^\nu\ln\Phi\Big) + 8(A^\mu\partial_\mu\ln\Phi)^2$$
$$+ 4(\nabla_\mu\partial_\nu\ln\Phi)A^\mu A^\nu + 4A^2(\partial_\mu\ln\Phi)^2 + (2+\beta)F_{\mu\nu}^2 + (\nabla_\mu A_\nu)^2 - (\nabla.A)^2$$
$$+ 4\nabla_\mu A_\nu\Big(A^\mu\partial^\nu\ln\Phi + A^\nu\partial^\mu\ln\Phi\Big) - 2A^\mu A^\nu\nabla_\mu A_\nu$$
$$- 4A^2A^\mu\partial_\mu\ln\Phi + \frac{1}{2}A^4\Big] - \frac{1}{2}\Phi_0^2\Big[(\partial_\mu\ln\Phi)^2 + \frac{1}{4}A^2 - A^\mu\partial_\mu\ln\Phi + \nu\Phi_0^4\Big]\Big\}. \tag{49}$$

To simplify[5] this action let us define $\Phi = \Phi_0 e^\varphi$ and $D_\mu\varphi \equiv \partial_\mu\varphi - \frac{1}{2}A_\mu$ which is actually gauge invariant since under the gauge transformations (4) $\varphi \to \varphi - \frac{1}{2}\zeta(x)$. Then (49) becomes

$$\tilde{S}_{NMG} = \int d^3x \sqrt{-g} \Bigg\{ \Phi_0^2 \Big[\sigma R - (8\sigma + \frac{1}{2})(D_\mu\varphi)^2 - \frac{1}{2}\nu\Phi_0^4 + (\frac{5}{2} + \beta)\Phi_0^{-4}F_{\mu\nu}^2 \Big]$$

$$+ \Phi_0^{-2} \Big[R_{\mu\nu}^2 - \frac{3}{8}R^2 + 4R_{\mu\nu}D^\mu\varphi D^\nu\varphi - 2R(D_\mu\varphi)^2 + 8(D_\alpha\varphi)^4 \tag{50}$$

$$+ 16(\nabla_\mu D_\nu\varphi)\partial^\mu\varphi\partial^\nu\varphi + 8D_\mu\varphi\partial^\mu\varphi\nabla\cdot A - 2A^2\nabla_\mu D_\mu\varphi \Big] \Bigg\}.$$

From this action, one can find the vacuum and study the excitations about it but this route, as we we noted in the text, is rather tedious compared to the Jordan frame action that we worked with. More importantly, the Einstein-frame action is not scale invariant and hence the idea, put forward above and in [12], that graviton becomes massive after the scale symmetry gets broken spontaneously (or radiatively) needs to be reinterpreted.

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[1] K. S. Stelle, Phys. Rev. D **16**, 953 (1977); Gen. Rel. Grav. **9**, 353 (1978).

[2] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett. **102**, 201301 (2009); Phys. Rev. D **79**, 124042 (2009).

[3] I. Gullu and B. Tekin, Phys. Rev. D **80**, 064033 (2009).

[4] S. Deser, Phys. Rev. Lett. **103**, 101302 (2009).

[5] M. Nakasone and I. Oda, Prog. Theor. Phys. **121**, 1389 (2009).

[6] Y. Liu and Y. W. Sun, Phys. Rev. D **79**, 126001 (2009).

[7] I. Gullu, T. C. Sisman and B. Tekin, Phys. Rev. D **81**, 104017 (2010).

[8] I. Gullu, T. C. Sisman and B. Tekin, Phys. Rev. D **83**, 024033 (2011).

[9] H. Ahmedov and A. N. Aliev, Phys. Rev. Lett. **106**, 021301 (2011).

[10] P. Horava, Phys. Rev. D **79**, 084008 (2009).

[11] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Annals Phys. **140**, 372 (1982).

[12] S. Dengiz and B. Tekin, Phys. Rev. D **84**, 024033 (2011).

[13] P. N. Tan, B. Tekin and Y. Hosotani, Phys. Lett. B **388**, 611 (1996); Nucl. Phys. B **502**, 483 (1997).

[14] S. R. Coleman and E. J. Weinberg, Phys. Rev. D **7**, 1888 (1973).

[15] I. Gullu, T. C. Sisman and B. Tekin, Class. Quant. Grav. **27**, 162001 (2010).

[16] I. Gullu, T. C. Sisman and B. Tekin, Phys. Rev. D **82**, 124023 (2010).

[17] P. Breitenlohner and D. Z. Freedman, Phys. Lett. B **115**, 197 (1982).

[18] A. R. Gover, A. Shaukat and A. Waldron, Nucl. Phys. B **812**, 424 (2009).

[19] A. Higuchi, Nucl. Phys. B **282**, 397 (1987).

[20] M. R. Tanhayi, S. Dengiz and B. Tekin, "Weyl-Invariant Higher Curvature Gravity Theories in n Dimensions," arXiv:1201.5068 [hep-th].

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